Lecture 1

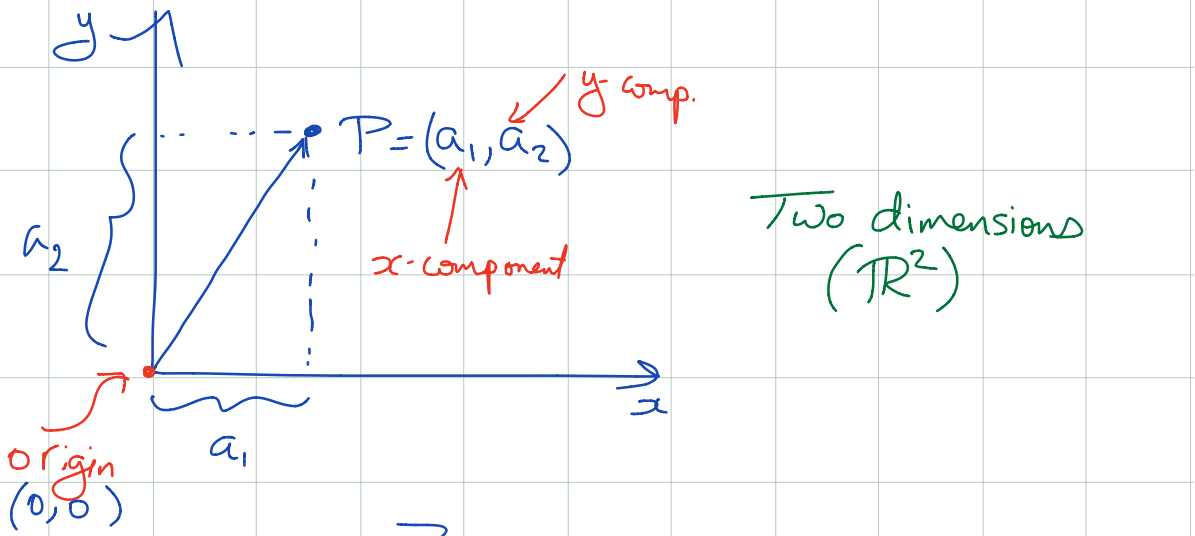
- We have already learned about differentiation (computing rates of change, tangents to curves, velocities...) & Integration (areas under curves, ...) and the relationship between Integration & differentiation (Fundamental theorem of calculus).
- But we were working with functions of one variable only!

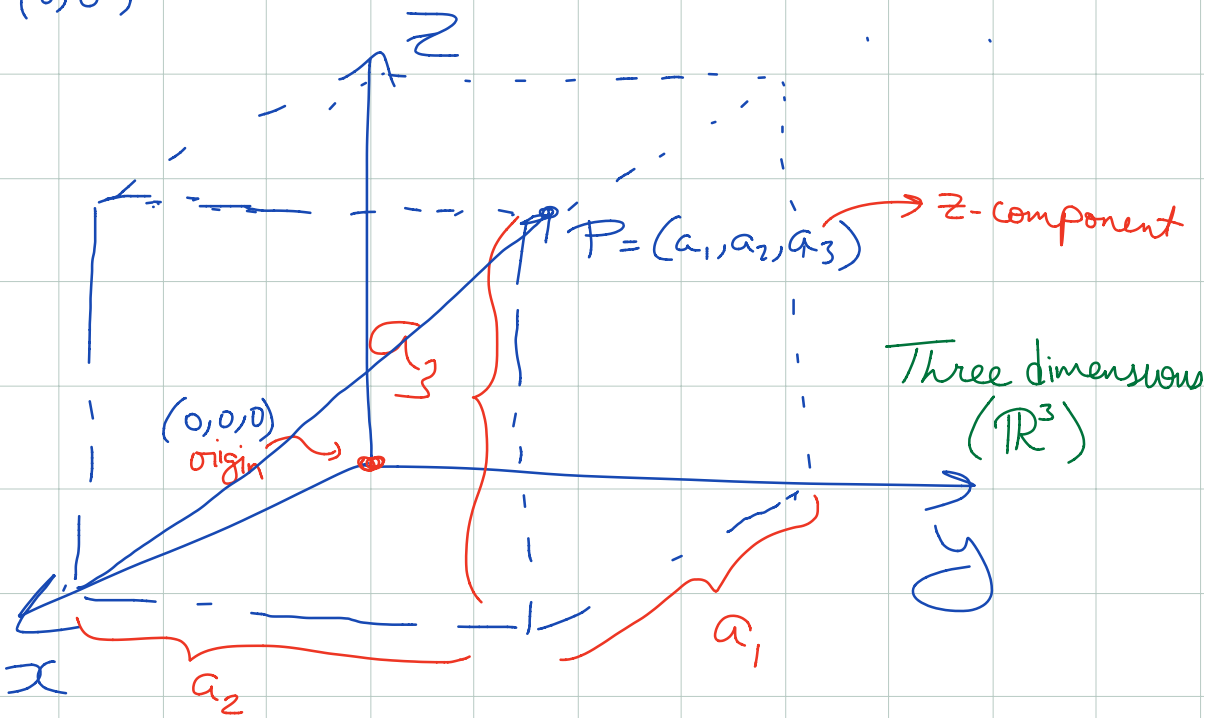
- We live in a 3-dimensional world (4, if you count time) and many interesting functions depend not just on one variable but many

- We would ultimately like to generalize differentiation & integration & understand

Their relationship & their applications when we have many variables.

1.1 Vectors in 2 & 3 dimensional Space





Vector addition & Scalar Multiplication

Define: $(a,b,c)+(d,e,f)=(a+d,b+e,c+f)$

So $(1,1,2)+(-1,7,\pi)=(0,8,2+\pi)$

- $(a_1,a_2,a_3)+\underbrace{(-a_1,-a_2,-a_3)}_{\text{negative of } (a_1,a_2,a_3)}=\underbrace{(0,0,0)}_{\text{zero element or zero vector or zero}}$

- **Define:** $\alpha(a,b,c)=(\alpha a,\alpha b,\alpha c)$ (α: scalar; (a,b,c): vector)

So $2(1,-7,5)=(2,-14,10)$

Properties:

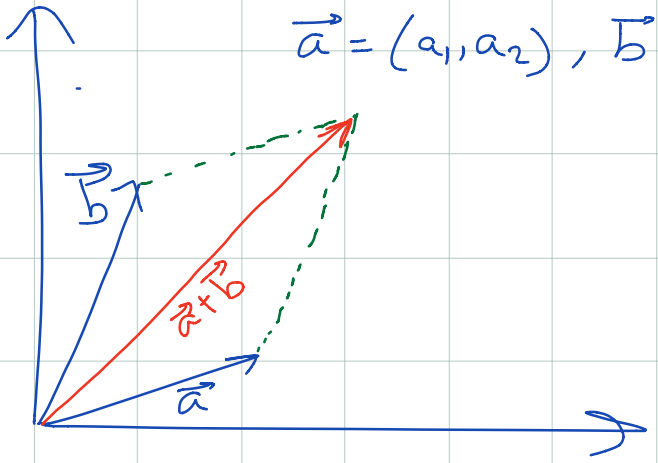
- $\alpha(0,0,0)=(0,0,0)$
- $0(a,b,c)=(0,0,0)$
- $1(a,b,c)=(a,b,c)$
- $(\alpha\beta)(a,b,c)=\alpha(\beta(a,b,c))$ ⟶ associativity

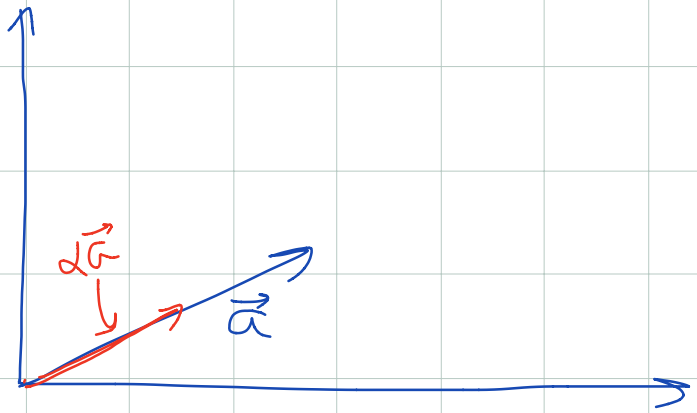
- $(\alpha+\beta)(a,b,c) = \alpha(a,b,c) + \beta(a,b,c)$ → distributivity
- $\alpha[(a_1,a_2,a_3) + (b_1,b_2,b_3)] = \alpha(a_1,a_2,a_3) + \alpha(b_1,b_2,b_3)$

(Proofs follow "easily" from the definitions)

Geometry of Vector Operations

$\vec{a} = (a_1,a_2)$, $\vec{b} = (b_1,b_2)$





Vectors: Directed line segments, with a beginning & end (↗). Translating a vector gives you the same vector

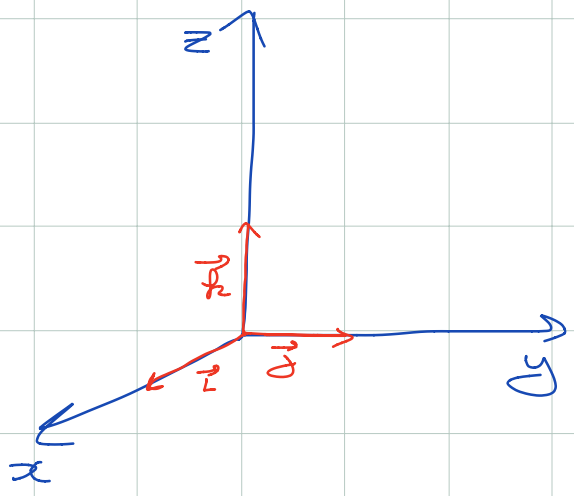
Standard Basis Vectors

It will be convenient to give names to the following vectors

$\vec{i} = (1,0,0)$

$\vec{j} = (0,1,0)$

$\vec{k} = (0,0,0)$



So: $(3,4,5) = 3(1,0,0) + 4(0,1,0) + 5(0,0,1)$

$= 3\vec{i} + 4\vec{j} + 5\vec{k}$

In general $(a,b,c) = a\vec{i} + b\vec{j} + c\vec{k}$.

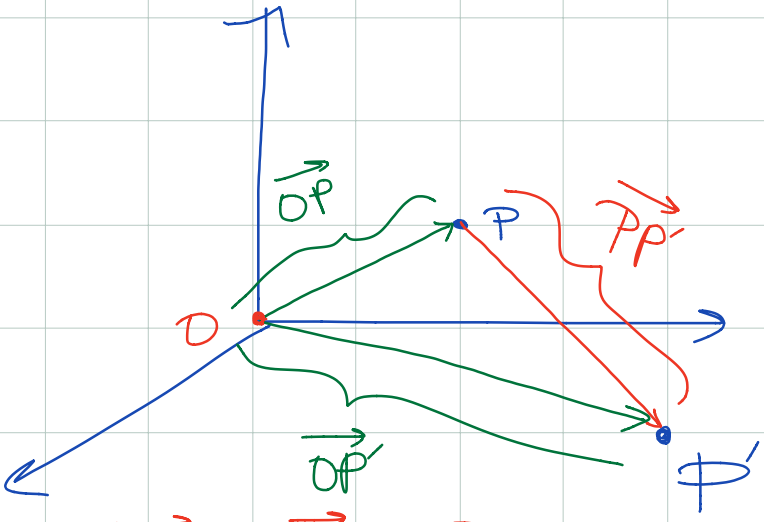
The vector joining 2 points

If the point P has coordinates (x, y, z)

and the point P′ has coordinates (x', y', z')

Then the vector $\overrightarrow{PP'}$ has components

$(x'-x, y'-y, z'-z)$



$$\overrightarrow{PP'} = \overrightarrow{OP'} - \overrightarrow{OP}$$

$$= \overrightarrow{OP'} + \overrightarrow{PO}$$

Example: add the vector from $(1,1,2)$ to $(2,2,3)$ to the vector from $(0,0,1)$ to $(-7,-5,-3)$

Solution: The first vector is $(2,2,3)-(1,1,2)=(1,1,1)$

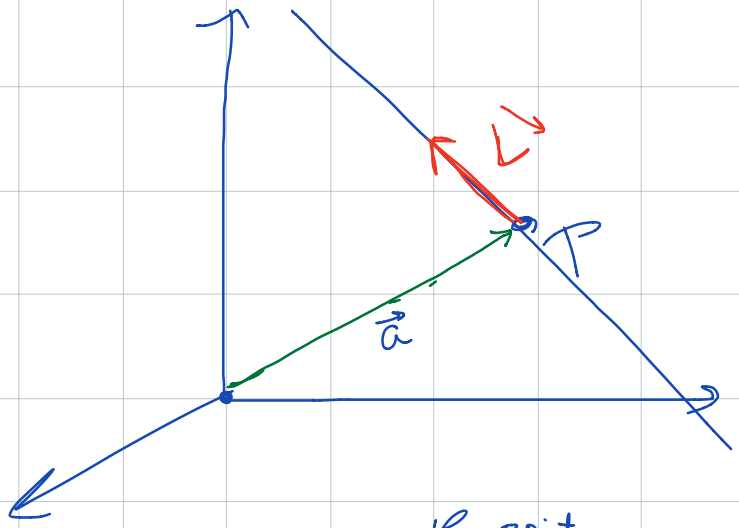
The second vector is $(-7,-5,-3)-(0,0,1)=(-7,-5,-4)$

So their sum is $(1,1,1)+(-7,-5,-4)=(-6,-4,-3)$.

$=-6\vec{i}-4\vec{j}-3\vec{k}$

Equations of Lines:

- Equations of a line ℓ passing through a point P with the direction of a vector $\vec{v}$.



We can see that the point $\vec{a}+\vec{v}$ is on the line ℓ

$\vec{a}+2\vec{v}$ is also on the line

$\vec{a}-0.5\vec{v}$ is also on ℓ

In fact, any point on ℓ is of the form $\vec{a}+t\vec{v}$ for some scalar t.

Point-Direction Form of a Line:

The equation of the line ℓ passing through the tip of $\vec{a} = (x_1, y_1, z_1)$ in the direction of $\vec{v} = (a, b, c)$ is

$$\vec{\ell}(t) = \vec{a} + t\vec{v}$$

In coordinate form, the equations are

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$$

Remark: When working in 2D we simply drop the z-coordinate.

Example: Find the equation of the line passing through the point $(1,1,2)$ in the direction $2\vec{i} - 3\vec{j} + 4\vec{k}$

Soln: $\vec{\ell}(t) = \vec{a} + t\vec{v}$ where $\vec{a} = (1,1,2)$ & $\vec{v} = (2,-3,4)$

So $\begin{cases} x = 1 + 2t \\ y = 1 - 3t \\ z = 2 - 4t \end{cases}$

Example: In what direction does the line

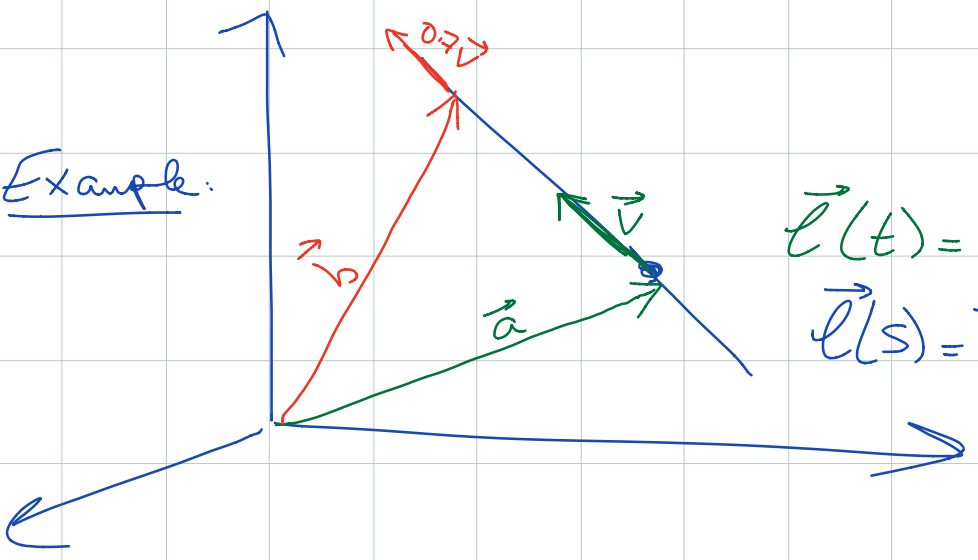
$$x = -3t + 2$$
$$y = -2t + 2$$
$$z = 8t + 2$$

Point?

answer: It points in the direction of $\vec{v} = (-3, -2, 8)$

Remark: Equation of a line is <u>not</u> unique

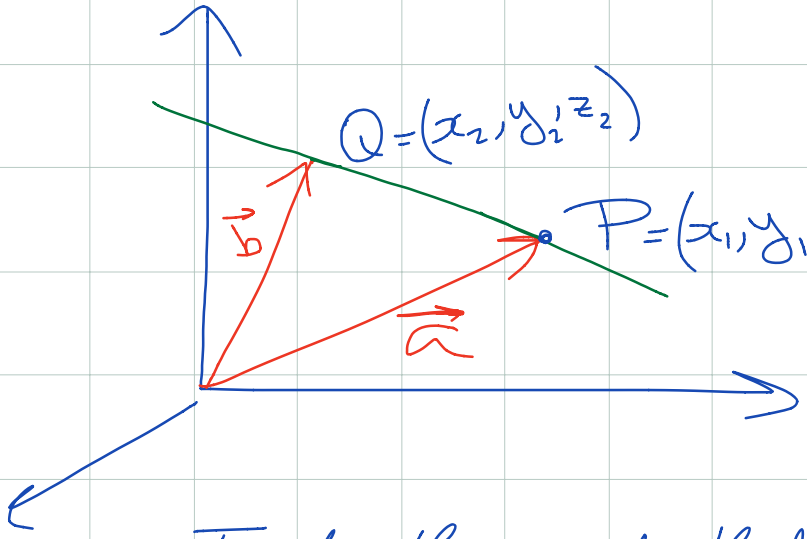
Example:



$$\vec{\ell}(t) = \vec{a} + t\vec{v}$$
$$\vec{\ell}(s) = \vec{b} + s\vec{v}$$

Point-point form of a line

Want an equation of a line passing through the endpoints of 2 vectors $\vec{a}$ & $\vec{b}$



To do this, note that the vector $\vec{b}-\vec{a}$ is in the direction of the line

So $$\vec{\ell}(t)=\vec{a}+(\vec{b}-\vec{a})t \quad (1)$$

$$\vec{\ell}(t)=(1-t)\vec{a}+t\vec{b}$$

Rewriting (1) in coordinate form:

$$x = x_1+(x_2-x_1)t$$
$$y = y_1+(y_2-y_1)t$$
$$z = z_1+(z_2-z_1)t$$

— Parametric equation of a line in point-point form

Remark: we can eliminate t to get

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

(provided the denominators are not zero)

Example: Find the equation of the line passing through $(4,5,6)$ & $(2,1,2)$

Soln:

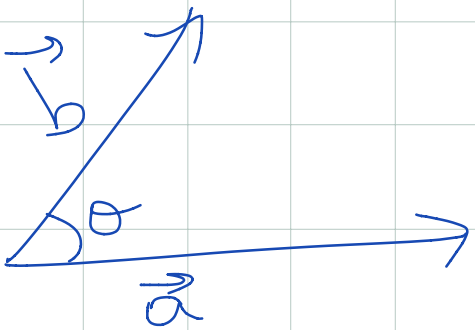
$$x = 4 + (2-4)t$$
$$y = 5 + (1-5)t$$
$$z = 6 + (2-6)t$$

or $\vec{\ell}(t) = (4,5,6) + t(-2,-4,-4)$

1.2 The inner product, Length & Distance

Say, you have two vectors $\vec{a}$ & $\vec{b}$

and you want to determine the angle between them



Inner products will help us do this

Define the inner product of $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ to be (the number)

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

↳ Sometimes we write $\langle \vec{a}, \vec{b} \rangle$

Example: Find the inner product of

$\vec{a} = 3\vec{i} + \vec{j} - 2\vec{k}$ & $\vec{b} = \vec{i} - \vec{j} + \vec{k}$

Sol'n: $\vec{a} \cdot \vec{b} = 3 \cdot 1 + 1 \cdot (-1) + (-2)(1) = 0$

Properties of Inner Products

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors in $\mathbb{R}^3$ & α, β be real numbers in $\mathbb{R}$. Then

(i) $\vec{a} \cdot \vec{a} \geq 0$
& $\vec{a} \cdot \vec{a} = 0$ if and only if $\vec{a} = 0$

(ii) $\alpha\, \vec{a} \cdot \vec{b} = \alpha(\vec{a} \cdot \vec{b})$
& $\vec{a} \cdot \alpha\vec{b} = \alpha(\vec{a} \cdot \vec{b})$

(iii) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

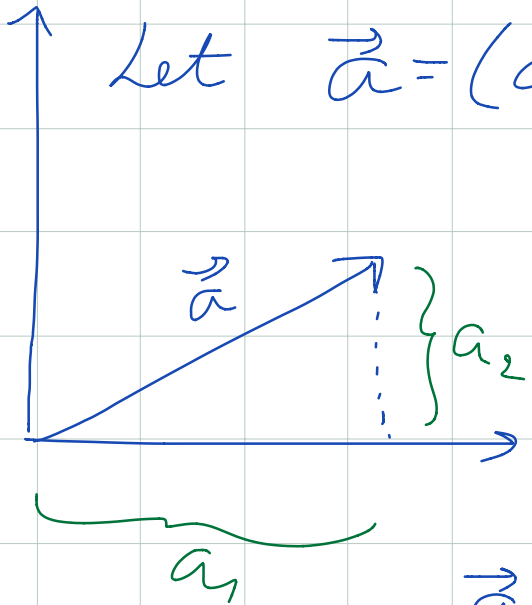
& $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

(iv) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Exercise: Try to prove these properties

The length/norm of a vector

Let $\vec{a} = (a_1, a_2)$



by Pythagoras's theorem

the length of $\vec{a}$ is $\sqrt{a_1^2 + a_2^2}$

But $\vec{a} \cdot \vec{a} = a_1 \cdot a_1 + a_2 \cdot a_2 = a_1^2 + a_2^2$

So $\vec{a} \cdot \vec{a} = (\text{length of } \vec{a})^2$

we write this as $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$\|\vec{a}\|$ = norm of $\vec{a}$

In 3D, if $\vec{a} = (a_1, a_2, a_3)$

$$\|\vec{a}\|^2 = \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = (\text{length of } \vec{a})^2$$

Unit Vectors: For any non-zero vector

$\frac{\vec{a}}{\|\vec{a}\|}$ is a unit vector (i.e. its length is 1)

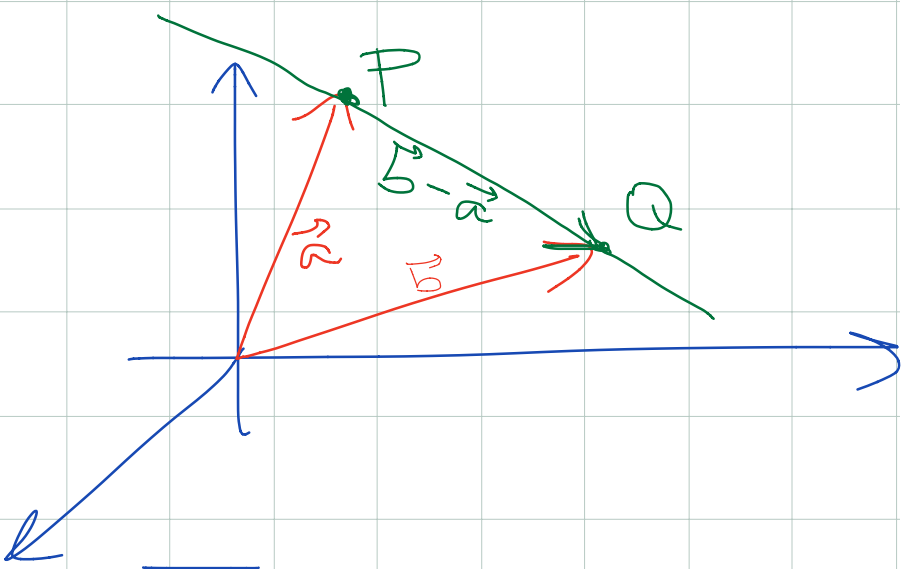
= make it unit length

Example: Normalize the vector

$\vec{v} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

answer: $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2\vec{i} + 3\vec{j} + 4\vec{k}}{\sqrt{2^2 + 3^2 + 4^2}} = \frac{2}{\sqrt{29}}\vec{i} + \frac{3}{\sqrt{29}}\vec{j} + \frac{4}{\sqrt{29}}\vec{k}$

Distance:



The distance between P & Q is $\|\vec{b}-\vec{a}\|$

Summary: Let $\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = (b_1, b_2, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then
- $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- $\|\vec{a}\| = \sqrt{a_1^2+a_2^2+a_3^2} = \sqrt{\vec{a}\cdot\vec{a}}$
- the vector $\frac{\vec{a}}{\|\vec{a}\|}$ is normalized
  i.e, it has unit norm
- the distance between the endpoints of $\vec{a}$ & $\vec{b}$ is $\|\vec{b}-\vec{a}\|$

The angle between 2 vectors

Theorem: Let $\vec{a}$ & $\vec{b}$ be two vectors in $\mathbb{R}^3$ & let θ where $0 \leq \theta \leq \pi$, be the angle between them. Then

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta.$$

Exercise: Read the proof in the book

Example: Find the angle between the vectors

$(1,1,2)$ & $(1,-1,1)$

Solution: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} = \frac{1\times1 + 1\times(-1) + 2\times1}{\sqrt{1^2+1^2+2^2} \times \sqrt{1^2+1^2+1^2}} = \frac{2}{\sqrt{18}}$$

$$= \frac{2}{3\sqrt{2}} \Rightarrow \theta = \arccos\left(\frac{2}{3\sqrt{2}}\right) = \cos^{-1}\theta.$$

Cauchy-Schwarz

Corollary: $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$

with equality if & only if $\vec{a}$ is a scalar multiple of $\vec{b}$ (or one of them is 0)

Proof: $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$

$\Rightarrow |\vec{a}\cdot\vec{b}| = \|\vec{a}\|\|\vec{b}\||\cos\theta|$

$\leq \|\vec{a}\|\|\vec{b}\|$ $(|\cos\theta| \leq 1)$

Moreover equality can only happen if $\vec{a}=0, \vec{b}=0$

or $\cos\theta = 0$ ∎

Remark: If $\vec{a}$ & $\vec{b}$ are nonzero, then

If $\vec{a}\cdot\vec{b} = \vec{0} \Rightarrow \cos\theta = 0 \Rightarrow \vec{a}$ & $\vec{b}$ are perpendicular

If $\vec{a}$ & $\vec{b}$ are perpendicular then $\cos\theta = 0$

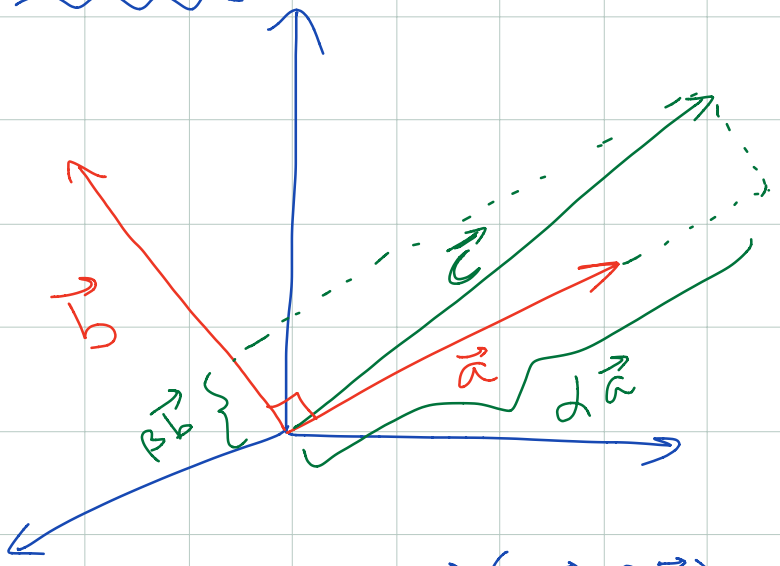
$\Rightarrow \vec{a}\cdot\vec{b} = 0$

Def'n: If $\vec{a}\cdot\vec{b} = 0$, we say they are orthogonal

Def'n: If $\vec{a}\cdot\vec{b} = 0$ & $\|\vec{a}\| = \|\vec{b}\| = 1$, we say $\vec{a}$ & $\vec{b}$ are orthonormal

Example: Let $\vec{a}$ & $\vec{b}$ be two orthogonal vectors. Let $\vec{c}$ be a vector in the plane spanned by $\vec{a}$ & $\vec{b}$. We can write $\vec{c} = \alpha\vec{a} + \beta\vec{b}$ for some scalars α & β. Use the inner product to determine α & β.

Solution:



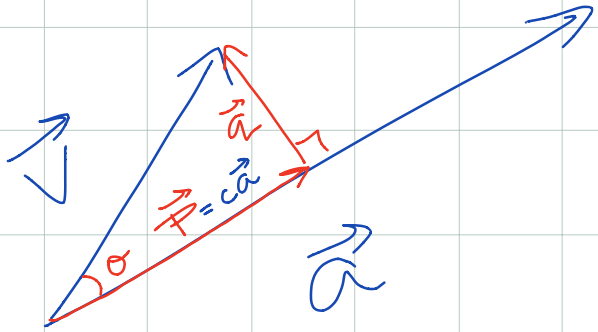
$$\vec{a}\cdot\vec{c} = \vec{a}\cdot(\alpha\vec{a}+\beta\vec{b}) = \alpha\,\vec{a}\cdot\vec{a} + \beta\underbrace{\vec{a}\cdot\vec{b}}_{=0}$$

$$\Rightarrow \alpha = \frac{\vec{a}\cdot\vec{c}}{\|\vec{a}\|^2}$$

Similarly $\beta = \dfrac{\vec{b}\cdot\vec{c}}{\|\vec{b}\|^2}$

We call $\alpha\vec{a}$ the projection of $\vec{c}$ along $\vec{a}$
& $\beta\vec{b}$ the projection of $\vec{c}$ along $\vec{b}$

Orthogonal Projection



$\vec{p}$ is the orthogonal projection of $\vec{v}$ on $\vec{a}$

$$\vec{v} = c\vec{a} + \vec{q}$$

$$\Rightarrow \vec{a}\cdot\vec{v} = c\,\vec{a}\cdot\vec{a} + \underbrace{\vec{a}\cdot\vec{q}}_{0}$$

$$\Rightarrow c = \frac{\vec{a}\cdot\vec{v}}{\vec{a}\cdot\vec{a}} \Rightarrow \vec{p} = \frac{\vec{a}\cdot\vec{v}}{\|\vec{a}\|^2}\vec{a}$$

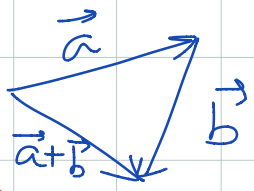
$\vec{p}$ is the orth. proj of $\vec{v}$ on $\vec{a}$

Example: The orthogonal proj. of $\vec{i}+\vec{j}$ on $\vec{i}-2\vec{j}$ is

$$\vec{P} = \frac{(\vec{i}+\vec{j})\cdot(\vec{i}-2\vec{j})}{(\vec{i}-2\vec{j})\cdot(\vec{i}-2\vec{j})}(\vec{i}-2\vec{j}) = \frac{1-2}{1+4}\vec{i}-2\vec{j} = -\frac{1}{5}(\vec{i}-2\vec{j})$$

Triangle inequality

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$



Proof: $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$$= \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$

$$\leq \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$$

$$= (\|\vec{a}\| + \|\vec{b}\|)^2$$

$$\Rightarrow \|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$ ∎